Assignment – III

MATHEMATICS – III

SEMESTER-IV (CS/IT), Paper Code: M401

Algebraic Structures

1) Prove that a group (G,*) is commutative if and only if $(a*b)^2 = a^2 * b^2$, for all $a, b \in G$.
2) Show that in a group (G,*), $(a*b)^{-1} = b^{-1} * a^{-1}$, $\forall\, a, b \in G$.
3) Show that a group (G,*) is abelian iff $(a*b)^{-1} = a^{-1} * b^{-1}$, $\forall\, a, b \in G$.
4) Show that if every element of a group (G,*) be its own inverse, then it is an abelian group. Is the converse true?
5) Let G be a group. If $a, b \in G$ such that $a^4 = e$, the identity element of G and $ab = ba^2$, prove that $a = e$.
6) Let (G,*) be a group, $a^5 = e$ & $aba^{-1} = b^2$ for some $a, b \in G$. Find the order of b.
7) Let (G,*) be a group & $a \in G$, $o(a) = 24$. Find the order of a^4, a^7, a^{10} .
8) Show that for any two subgroups H and K of a group G, $H \cap K$ is also a subgroup of G.
9) The subset H of a group $(G, \circ)$ is defined by H = $\{ x \in G : x \circ g = g \circ x \;\forall g \in G \}$. Prove that H is a subgroup of G.
10) Define a cyclic group. Prove that every cyclic group is abelian (commutative).
11) Show that the group $(Z_5, +)$, i.e. the additive group of all integers modulo 5 is cyclic. Find all generators of Z_5 .
12) Prove that the order of each subgroup of a finite group is a divisor of the order of the group. (Lagrange's Theorem)
13) Let H be a subgroup of a group G. Show that H is normal in G if and only if $xhx^{-1} \in H$ $\forall h \in H, \forall x \in G$.
14) Define normal subgroup of a group. If G is a group and H is a subgroup of index 2 in G, prove that H is a normal subgroup of G.
15) Let G = $\left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : ad \neq 0 \right\}$ and H = $\left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right\}$. Prove that H is a normal subgroup of the group $G, \cdot$
16) Let A = $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}$, B = $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & 5 & 2 \end{pmatrix}$ be two permutations. Show that $AB \neq BA$.
17) Let H be a normal subgroup of a group G and G/H be the set of all cosets of H in G. Show that G/H forms a group under the composition

$$(aH).(bH) = (ab)H \text{ for all } a,b \in G.$$

18) Let $f : G \rightarrow G'$ be a groups of homomorphism. Show that f is one-one if and only if ker(f) ={e}, where e is a unit element of G.

19) Let f : G $\to G'$ be a groups of homomorphism. Show that (i) Imf is a subgroup of G' and (ii)Kernel of f is a normal subgroup of G.

20) Let R be the additive group of real numbers and C^* be the multiplicative group of nonzero complex numbers. If $f: R \to C^*$ is a group of homomorphism defined by $f(x) = e^{2\pi ix}$ for all $x \in R$, find the kernel of f.

21) Let (R,+) be the group of real numbers under addition and $(R^+,\cdot)$ be the group of positive real numbers under multiplication. Define $f: R \to R^+$ by $f(a) = e^a \ \forall\, a \in R$. Show that f is an isomorphism from (R,+) onto $(R^+,\cdot)$.

22) Let (Q,+) be the additive group of rational numbers and $(Q^+,\cdot)$ be the multiplicative group of positive rational numbers. Are these two groups isomorphic?

23) Prove that the set of all even integers form a commutative ring.

24) Prove that the intersection of two subrings is a subring.

25) If in a ring R with unity, $(xy)^2 = x^2y^2$ for all $x, y \in R$, then show that R is commutative.

26) Prove that the ring of matrices of the form $\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$ of real numbers is a field.

27) Prove that every finite integral domain is a field.

28) Let R and S be two rings and $f: R \to S$ be a ring homomorphism. Show that kernel of f is a subring of R.